Supplemental Material: Semi-Implicit Pairwise Descent for Nonlocal Continuum Mechanics

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A Proof of Theorem 4.1

Consider a tetrahedron with four vertices $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. A linear shape function defined on vertex 0 is formulated as

$$N_0(x, y, z) = a + bx + cy + dz, \tag{1}$$

and its gradient is $\nabla N_0 = (b, c, d)$.

Applying the Kronecker-δ property $N_i(\mathbf{x}_j) = \delta_{ij}$, we have:

$$\begin{bmatrix} x_1 - x_0 & y_2 - y_1 & z_2 - z_1 \\ x_2 - x_0 & y_3 - y_1 & z_3 - z_1 \\ x_3 - x_0 & y_4 - y_1 & z_4 - z_1 \end{bmatrix} \begin{bmatrix} b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \tag{2}$$

Multiplying both sides with the diagonal matrix $\mathrm{diag}(\boldsymbol{\xi}_{01})$, $\mathrm{diag}(\boldsymbol{\xi}_{02})$ and $\mathrm{diag}(\boldsymbol{\xi}_{03})$, and then summing up each row results in $\mathbf{K}\nabla N_0 = -\sum_{j=1,2,3} \boldsymbol{\xi}_{0j}$, where $\mathbf{K} = \xi_{01}\xi_{01}^T + \xi_{02}\xi_{02}^T + \xi_{03}\xi_{03}^T$. Move $\mathbf{K}$ to the right hand side proves $\nabla N_0 = -\mathbf{K}^{-1} \sum_{j=1,2,3} \boldsymbol{\xi}_{0j}$.

B Calculating the weights

After left-multiplying both sides of $\sum_{j\in\mathcal{V}_e} \mathbf{K}_e^{-1}\boldsymbol{\xi}_{ij} = \sum_{j\in\mathcal{V}_e} w_{ij,e}\boldsymbol{\xi}_{ij}$ with the shape tensor $\mathbf{K}_e$, we obtain:

$$\begin{aligned} \sum_{j\in\mathcal{V}_e} \boldsymbol{\xi}_{ij} = \mathbf{K}_e \sum_{j\in\mathcal{V}_e} w_{ij,e}\boldsymbol{\xi}_{ij} &= \sum_{j\in\mathcal{V}_e} w_{ij,e} \left(\sum_k \boldsymbol{\xi}_{ik} \otimes \boldsymbol{\xi}_{ik}\right) \boldsymbol{\xi}_{ij} \\ &= \sum_{j\in\mathcal{V}_e} w_{ij,e} \sum_k \left(\boldsymbol{\xi}_{ij} \cdot \boldsymbol{\xi}_{ik}\right) \boldsymbol{\xi}_{ik} \\ &= \sum_{j\in\mathcal{V}_e} \left(\sum_k w_{ik,e} \left(\boldsymbol{\xi}_{ik} \cdot \boldsymbol{\xi}_{ij}\right)\right) \boldsymbol{\xi}_{ij} \end{aligned}, \tag{3}$$

To guarantee the above equation stands for all linearly independent vectors of $\boldsymbol{\xi}_{ij}$, the following condition must be satisifed

$$\sum_k w_{ik,e} \left(\boldsymbol{\xi}_{ik} \cdot \boldsymbol{\xi}_{ij}\right) = 1. \tag{4}$$

Therefore, calculating the values for w_{ij} is equivalent to solving a linear system that contains 3 linear equations in 3 variables. By taking vertex 0 as an example, the linear system is written as

$$\underbrace{\begin{bmatrix} \xi_{01}\cdot\xi_{01} & \xi_{01}\cdot\xi_{02} & \xi_{01}\cdot\xi_{03} \\ \xi_{02}\cdot\xi_{01} & \xi_{02}\cdot\xi_{02} & \xi_{02}\cdot\xi_{03} \\ \xi_{03}\cdot\xi_{01} & \xi_{03}\cdot\xi_{02} & \xi_{03}\cdot\xi_{03} \end{bmatrix}}_{\mathbf{G}_0} \underbrace{\begin{Bmatrix} w_{01} \\ w_{02} \\ w_{03} \end{Bmatrix}}_{\mathbf{w}_0} = \underbrace{\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}}_{\mathbf{1}} \tag{5}$$

Note the coefficient matrix $\mathbf{G}_0$ is commonly referred to as the Gram matrix, whose determinant equals the square of the volume of the tetrahedron spanned by the three vectors ξ_{01}, ξ_{02} and ξ_{03}. Another important property is that $\mathbf{w}_0$ is positive definite; therefore, $\mathbf{w}_0$ can be uniquely determined as $\mathbf{w}_0 = \mathbf{G}_0^{-1}\mathbf{1}$.

To calculate the weighting coefficient $\mathbf{w}_j$ for other vertices, a straightforward solution is to follow the same procedure to construct the Gram matrix and calculate the inverse for each vertex, respectively. However, we propose a more efficient alternative by first reformulating $\mathbf{G}_0 = \mathbf{D}_0^T\mathbf{D}_0$, where $\mathbf{D}_0 = [\xi_{01}, \xi_{02}, \xi_{03}]$. The Gram matrix for other vertices can actually be constructed by utilizing the rotational symmetry of the tetrahedron vertices and affine transformation relations:

$$\begin{aligned} \mathbf{D}_1 &= \mathbf{D}_0 \begin{bmatrix} -\mathbf{e}_0 & \mathbf{e}_1 - \mathbf{e}_0 & \mathbf{e}_2 - \mathbf{e}_0 \end{bmatrix} \\ \mathbf{D}_2 &= \mathbf{D}_0 \begin{bmatrix} -\mathbf{e}_1 & \mathbf{e}_0 - \mathbf{e}_1 & \mathbf{e}_2 - \mathbf{e}_1 \end{bmatrix} \\ \mathbf{D}_3 &= \mathbf{D}_0 \begin{bmatrix} -\mathbf{e}_2 & \mathbf{e}_0 - \mathbf{e}_2 & \mathbf{e}_1 - \mathbf{e}_2 \end{bmatrix} \end{aligned} \tag{6}$$

where $\mathbf{e}_{0,1,2}$ are the basis vectors. After substituting the above equations into the corresponding linear system, which is analogous to Equation 5, we find that the weighting coefficients for the remaining vertices can be computed directly through simple algebraic operations on $\mathbf{G}_0^{-1}$. Consequently, the weighting coefficients for all vertices of a tetrahedron can be expressed as follows:

$$\begin{bmatrix} 0 & w_{10} & w_{20} & w_{30} \\ w_{01} & 0 & w_{21} & w_{31} \\ w_{02} & w_{12} & 0 & w_{32} \\ w_{03} & w_{13} & w_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & w_{01} & w_{02} & w_{02} \\ w_{01} & 0 & -G_{01}^{-1} & -G_{02}^{-1} \\ w_{02} & -G_{10}^{-1} & 0 & -G_{12}^{-1} \\ w_{03} & -G_{20}^{-1} & -G_{21}^{-1} & 0 \end{bmatrix}, \tag{7}$$

where G_{ij}^{-1} represents the (i, j)-th element of the matrix $\mathbf{G}_0^{-1}$. Since the matrix $\mathbf{G}_0^{-1}$ is symmetric, the weighting coefficient matrix is also symmetric, and thus $w_{ij,e} = w_{ji,e}$.

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C Relation to Classical Contact and Friction Constraints

C.1 Normal Contact

The VBE formulation does not introduce a new contact measure; instead, it represents the conventional normal distance as the stretching of a virtual element. Consider an active vertex–face pair, and let the closest point on the triangle be

$$\bar{\mathbf{y}} = \sum_{a=1}^{3} \beta_a \mathbf{y}_a, \qquad \sum_{a=1}^{3} \beta_a = 1, \tag{8}$$

where β_a are its barycentric coordinates. The signed normal distance and its distance constraint are

$$d_n = \mathbf{n}^T(\mathbf{y}_0 - \bar{\mathbf{y}}), \qquad c_n(\mathbf{y}) = d_n - \hat{d}. \tag{9}$$

For the same pair, our normal VBE defines

$$\mathbf{F}_n = \mathbf{I} + \left(\frac{d_n}{\hat{d}} - 1\right)\mathbf{n}\mathbf{n}^T, \qquad \mathbf{F}_n\mathbf{n} = \frac{d_n}{\hat{d}}\mathbf{n}. \tag{10}$$

Consequently, the VBE normal strain is simply a normalized representation of the same distance constraint:

$$C_n = \|\mathbf{F}_n\mathbf{n}\| - 1 = \frac{d_n - \hat{d}}{\hat{d}} = \frac{c_n}{\hat{d}}. \tag{11}$$

The total VBE energy of an active pair can therefore be written as

$$E_n = V_e\Psi_n = \frac{1}{2}V_e\kappa C_n^2 = \frac{1}{2}\underbrace{\frac{V_e\kappa}{\hat{d}^2}}_{k_n} c_n^2, \tag{12}$$

With $k_n = V_e\kappa/\hat{d}^2$, this is exactly the conventional quadratic distance-penalty energy. Since the two energies are the same scalar function of the nodal positions, their derivatives produce the same contact force:

$$-\nabla_{\mathbf{y}}E_n = -k_n c_n \nabla_{\mathbf{y}} c_n, \tag{13}$$

and, for a fixed contact normal,

$$\nabla_{\mathbf{y}_0} c_n = \mathbf{n}, \qquad \nabla_{\mathbf{y}_a} c_n = -\beta_a\mathbf{n}, \quad a = 1, 2, 3. \tag{14}$$

Thus, the VBE changes only the representation of the distance-based penalty model: it preserves both its energy and the barycentric distribution of its nodal forces. The same construction applies to an edge–edge pair after its two closest points are expressed using barycentric coordinates along the edges.

C.2 Friction

The same idea is used for friction by representing tangential relative displacement as virtual shear. Let the tangential relative displacement of the contact pair be

$$\mathbf{u}_t = (\mathbf{I} - \mathbf{n}\mathbf{n}^T)(\mathbf{y}_0 - \bar{\mathbf{y}}), \qquad d_t = \|\mathbf{u}_t\|, \qquad \mathbf{t} = \frac{\mathbf{u}_t}{d_t}. \tag{15}$$

The tangential VBE is defined by

$$\mathbf{F}_t = \mathbf{I} + \frac{d_t}{\hat{d}}\mathbf{t}\mathbf{n}^T, \tag{16}$$

and hence

$$C_t = \|(\mathbf{F}_t - \mathbf{I})\mathbf{n}\| = \frac{d_t}{\hat{d}}. \tag{17}$$

Its friction energy is therefore

$$E_t = V_e\mu\lambda f(C_t) = V_e\mu\lambda f\left(\frac{d_t}{\hat{d}}\right). \tag{18}$$

This representation allows elasticity, normal contact, and friction to enter the same pairwise assembly and SIPD solve.

Our current friction implementation does not explicitly model the transition between sticking and sliding, which may reduce its accuracy in scenarios dominated by static friction.

D Convergence of the Analytical Decomposition

We show that the analytical projection strategy in Section 5.1 produces a descent direction and, when combined with line search, ensures convergence. Let

$$\mathbf{Q}_e^k = (\mathbf{P}\mathbf{F}^{-1})_e^k = \mathbf{Q}_e^{k,+} + \mathbf{Q}_e^{k,-}, \tag{19}$$

For isotropic hyperelastic materials, $\mathbf{Q} = \mathbf{P}\mathbf{F}^{-1}$ is symmetric. The analytical decomposition preserves this symmetry and separates $\mathbf{Q}_e^k$ into positive and negative semi-definite parts satisfying

$$\mathbf{Q}_e^{k,+} \succeq \mathbf{0}, \qquad \mathbf{Q}_e^{k,-} \preceq \mathbf{0}. \tag{20}$$

Here, $\succeq \mathbf{0}$ and $\preceq \mathbf{0}$ denote positive and negative semi-definiteness, respectively. Together with $w_{ij,e}^+ \geq 0$, $w_{ij,e}^- \leq 0$, and $V_e > 0$, Equation 20 gives

$$\mathbf{L}_{ij,e}^{k,+} = V_e\left(w_{ij,e}^+\mathbf{Q}_e^{k,+} + w_{ij,e}^-\mathbf{Q}_e^{k,-}\right) \succeq \mathbf{0}. \tag{21}$$

Indeed, both terms in the parentheses are positive semi-definite: the first is a nonnegative scalar multiple of a positive semi-definite matrix, while the second is a nonpositive scalar multiple of a negative semi-definite matrix. Summation over the incident elements therefore preserves positive semi-definiteness, so $\mathbf{L}_{ij}^{k,+} = \sum_e \mathbf{L}_{ij,e}^{k,+} \succeq \mathbf{0}$.

For vertex i, define the local update matrix

$$\mathbf{B}_i^k = \frac{m_i}{h^2}\mathbf{I} + \sum_j \mathbf{L}_{ij}^{k,+}. \tag{22}$$

Since m_i and h are positive, Equation 21 implies $\mathbf{B}_i^k \succ \mathbf{0}$. Consequently, its inverse exists and is also positive definite. The SIPD direction is

$$\mathbf{d}_i^k = -(\mathbf{B}_i^k)^{-1}\nabla_i\mathcal{L}(\mathbf{y}^k). \tag{23}$$

Let $\mathbf{B}_k = \mathrm{diag}(\mathbf{B}_1^k, \ldots, \mathbf{B}_n^k)$ and assemble the local directions into $\mathbf{d}^k = -\mathbf{B}_k^{-1}\nabla\mathcal{L}(\mathbf{y}^k)$. Whenever $\nabla\mathcal{L}(\mathbf{y}^k) \neq \mathbf{0}$,

$$\begin{aligned}\nabla\mathcal{L}(\mathbf{y}^k)^T\mathbf{d}^k &= -\nabla\mathcal{L}(\mathbf{y}^k)^T\mathbf{B}_k^{-1}\nabla\mathcal{L}(\mathbf{y}^k) \\ &< 0,\end{aligned} \tag{24}$$

and hence the analytical decomposition produces a strict descent direction whenever the gradient is nonzero.

Since $\mathbf{B}_k$ is positive definite, Equation 24 shows that every SIPD update is a descent direction. Combined with line search, this ensures convergence.