Supplementary Material to “Towards a non-invasive diagnosis of portal hypertension based on an Eulerian CFD model with diffuse boundary conditions”

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1 Numerical Implementation

At the beginning of simulation, both the velocity and pressure fields are initialized to be zero. The transient term $\rho \frac{\partial \boldsymbol{v}}{\partial t}$ will be temporarily integrated into the momentum equation to update the velocity field. Therefore, the following governing equations will be solved in our numerical implementation

$$
\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + \mu \Delta \boldsymbol{v},
$$

$$
\nabla \cdot \boldsymbol{v} = 0.
$$

Note when both the velocity and pressure fields converge to their final solutions, the transient term $\rho \frac{\partial \boldsymbol{v}}{\partial t}$ will finally converge to 0. To solve above equations, the semi-implicit method for pressure-linked equations (SIMPLE) algorithm is applied and its full procedure is as follows [1]

1) Initialize both pressure field $p^*$ and the velocity field $\boldsymbol{v}^*$ to be zero;
2) Solve the following momentum equation to obtain $\boldsymbol{v}'$

$$
\rho \frac{\boldsymbol{v}' - \boldsymbol{v}^*}{\delta t} = -\nabla p^* + \mu \nabla^2 \boldsymbol{v}',
$$

where $\delta t$ is the time step size.
3) Solve the following pressure Poisson equation (PPE) to obtain $p'$.

$$
\nabla \cdot \frac{\delta t}{\rho} \nabla p' = \nabla \cdot \boldsymbol{v}',
$$

4) Add $p'$ to $p^*$;
5) Return to step 2 and repeat the whole procedure until the converged solution is obtained
6) Set $\boldsymbol{v}^* = \boldsymbol{v}'$, return to step 2 and repeat until the transient term vanishes.
1.1 Discretization of the momentum equation

For simplicity, we only give the discretized formulations for a one-dimensional problem. Consider a cell that is located inside the vessel but far away from the boundary, its momentum equation can be discretized as

\[
\frac{u'_{i} - u^*_i}{\delta t} = \frac{\mu}{\rho} \left( \frac{u'_{i+1} - u'_{i} - 2u'_{i} + u'_{i-1}}{d^2} \right) - \frac{1}{\rho} \left( \frac{p^*_i - p^*_{i-1}}{d} \right).
\]

(4)

![Diagram](a)

Fig. 1. Discretization. (a) discretization of the momentum equation; (b) discretization of the pressure Poisson equation.

When the cell is near the boundary, two different cases should be considered. As shown in Fig. 1(a), if the boundary is located between \(u_i\) and \(p_i\), the discretized momentum equation is written as

\[
\frac{u'_{i} - u^*_i}{\delta t} = \frac{\mu}{\rho} \left( \frac{u'_{i+1} - 2\frac{A_i}{d} u'_{i-1} - 2u'_{i} + u'_{i+1}}{d^2} \right) - \frac{D_i}{\rho} \left( \frac{p^*_i - p^*_{i-1}}{d} \right),
\]

\[
A_i = \frac{2d}{d + \phi_i^p}, \quad D_i = \frac{2d}{d + 2\phi_i^p}.
\]

(5)

Otherwise, if the boundary is located between \(p_i\) and \(u_{i+1}\), the discrete momentum equation is written as

\[
\frac{u'_{i} - u^*_i}{\delta t} = \frac{\mu}{\rho} \left( \frac{u'_{i+1} - 2\frac{A_i}{d} u'_{i-1} - 2u'_{i} + u'_{i+1}}{d^2} \right) - \frac{1}{\rho} \left( \frac{p^*_i - p^*_{i-1}}{d} \right),
\]

\[
A_i = \frac{2d}{d + \phi_i^u}.
\]

(6)

After assembling all equations, the conjugate gradient least squares (CGLS) algorithm in Eigen was applied to solve the linear system of equations.

1.2 Discretization of the pressure Poisson equation

If a cell is located inside and far away from the vessel, the discretized pressure Poisson equation can be written as

\[
\frac{\delta t p'_{i+1} - 2p'_{i} + p'_{i-1}}{\rho} = \frac{u^*_{i+1} - u^*_{i}}{d},
\]

(7)
When the cell is located near the boundary, its formulation depends on the location of the boundary. If the boundary is located between $p_i$ and $u_{i+1}$, the discrete pressure Poisson equation is written as

$$
\frac{\delta t}{\rho} B_i p_b + \left(2 - B_i\right) p'_{i-1} - 2 p'_i = C_i \frac{u_b - u'_i}{d},
$$

$$
B_i = \frac{2d}{d + \phi^*_i}, \quad C_i = \frac{2d}{d + 2\phi^*_i},
$$

(8)

Otherwise, if the boundary is located between $p_i$ and $u_{i+1}$, the discretized pressure Poisson equation is written as

$$
\frac{\delta t}{\rho} B_i p_b + \left(2 - B_i\right) p'_{i-1} - 2 p'_i = \frac{u'_{i+1} - u'_i}{d},
$$

$$
B_i = \frac{2d}{d + \phi^*_i},
$$

(9)

The linear system of equations can be solved with the CGLS algorithm as well.

References